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Remarks on Undecidability, Incompleteness and the Integrability Problem

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Abstract In their seminal paper *Undecidability and incompleteness in classical mechanics* (Int. J. Theor. Phys. 30:1041–1073, 1991), N.C.A. da Costa and F.A. Doria introduced a powerful method for studying the appearance of undecidability and incompleteness in mathematics and theoretical physics. In this work their results are applied to integrability theory. Specifically, it is pointed out that it is not possible to expect the existence of an algorithm able to decide whether a given partial differential equation is integrable or not.

Keywords Integrable equations · Generalized symmetries · Undecidability · Incompleteness · Algorithms

1 Introduction

This work is on the "integrability problem" for partial differential equations. Integrability will be understood as s-*integrability*, following [6, 13, 17] and specially Peter Olver's [18]: a partial differential equation (or more generally, a system of PDEs) is s-*integrable* if and only if it possesses an infinite number of generalized symmetries, a notion to be tersely reviewed below (intuitively, a generalized symmetry is an infinitesimal deformation of the dependent variables which transforms solutions into solutions to first order in the deformation parameter, see [18]). It appear that the reasons for introducing such a definition are largely experimental: extensive lists of s-integrable equations (and systems) have been compiled (in some cases even complete classifications are known, see [13–15, 17–21, 27, 28] and references therein) and it has been observed that equations in these lists possess *at least one* of the following appealing characteristics: they are linearizable; they possess an infinite

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number of conservation laws; they admit Bäcklund transformations; they can be integrated via scattering/inverse scattering. It is also important to stress that s-integrability (hereafter just "integrability") *is not* simply an interesting technical characteristic of some partial differential equations: integrable equations appear naturally in very many fundamental physical theories and in other parts of mathematics [1, 22, 29]. For example, the well-known Korteweg-de Vries (KdV) equation

$$u_t = u_{xxx} + u u_x, \tag{1}$$

is integrable (see [18] and references therein). It was considered originally in connection with shallow water waves [1, 22], but it also appears in topological field theory [12] and in Riemannian geometry of the Virasoro group [16].

Now, due to their ubiquitousness, physical relevance, and undoubted mathematical interest, it is quite natural to investigate the possibility of classifying integrable equations. Would it be possible to check *algorithmically* whether an equation is integrable or not? Would it be possible to *prove formally* that an integrable equation is indeed integrable? In this article I announce the following two results: First, such an algorithm does not, and can not, exist. Second, there exists an integrable equation for which its integrability *cannot be proven* within a standard axiomatic system for mathematics such as the Zermelo–Fraenkel plus Axiom of Choice (ZFC) set theory.

This work is based on the deep results obtained by Newton da Costa and Francisco Doria [7–9] on the incompleteness and undecidability phenomenon in mathematics and natural sciences. Roughly speaking, they found a way to translate the famous theorems on incompleteness and undecidability of formal arithmetic due to Kurt Gödel [2] into analytic statements, and this translation implies, in particular, negative answers to the questions above. I stress that the *existence* of the "da Costa–Doria translation" is really an important and deep fact, not just a "straightforward translation of well-known undecidability results": Gödel's famous undecidable statements do not have immediate mathematical meaning, and until well advanced the XXth Century it was believed, or at least hoped, that incompleteness and undecidability remain far from standard mathematics. This view was shaken in 1977 when a true statement in finitary combinatorics which is not provable in Peano arithmetic was found (see the chapter by J. Paris and L. Harrington in [2]) and then in 1991, when N. da Costa and F. Doria [7] realized that the existence of algorithmically unsolvable results in the theory of diophantine equations yields undecidability results in analysis and classical mechanics. Thus, the da Costa–Doria translation brings undecidability and incompleteness from the mathematical logic suburbs to the very center of mathematical and physical practice.

Full accounts of some of the results announced here appear in the longer work [25].

2 Symmetries and Integrability

This section is a short review of s-integrability, based primarily on the standard treatise by P. Olver [18]. The following conventions will apply: Independent variables will be denoted by x^i, $i = 1, 2, \ldots, n$, and dependent variables by u^α, $\alpha = 1, 2, \ldots, m$; the space E of independent and dependent variables will be simply an open subset of the product $\mathbf{R}^n \times \mathbf{R}^m$; k-tuples $J = (j_1, \ldots, j_k)$, $0 \le j_1, j_2, \ldots, j_k \le n$ will denote multi-indices of order $\#J = k$; and derivatives of u^α will be represented by sub-indices, so that if $J = (j_1, \ldots, j_k)$, then $u^\alpha_J = \partial^k u^\alpha / \partial x_{j_1} \ldots \partial x_{j_k}$.

2.1 Generalized Symmetries

A *generalized vector field* on E is a first order operator of the form

$$V=\sum_{i=1}^{n}\xi^{i}\frac{\partial}{\partial x^{i}}+\sum_{\alpha=1}^{m}\varphi^{\alpha}\frac{\partial}{\partial u^{\alpha}},\tag{2}$$

in which the functions ξ^i and φ^α are smooth functions of the independent variables x^i, the dependent variables u^α and arbitrary (but finite) numbers of partial derivatives u^α_J. The *infinite prolongation* of V is the formal operator

$$pr\,V=\sum_{i=1}^{n}\xi^{i}\frac{\partial}{\partial x^{i}}+\sum_{\alpha=1}^{m}\sum_{\#J\geq 0}\varphi^{\alpha}_{J}\frac{\partial}{\partial u^{\alpha}_{J}},\tag{3}$$

in which the functions φ^α_J are obtained inductively by means of the relation

$$\varphi^{\alpha}_{J\,i}=D_i\,\varphi^{\alpha}_{J}-\sum_{k=1}^{n}D_i(\xi^{k})\,u^{\alpha}_{J\,k};\qquad \varphi^{\alpha}_{0}=\varphi^{\alpha},\tag{4}$$

and the *total derivatives* D_i are defined as

$$D_i=\frac{\partial}{\partial x^{i}}+\sum_{\alpha=1}^{m}\sum_{\#J\geq 0}u^{\alpha}_{J\,i}\frac{\partial}{\partial u^{\alpha}_{J}}.\tag{5}$$

Definition 1 A generalized vector field V is a generalized symmetry of the system $\Xi_a=0$ if and only if

$$pr\,V(\Xi_a)=0\tag{6}$$

whenever $u^\alpha(x^i)$ is a solution of $\Xi_a=0$.

It is enough [18] to consider generalized symmetries of the form

$$V=\sum_{\alpha=1}^{m}G^{\alpha}\frac{\partial}{\partial u^{\alpha}}.\tag{7}$$

In this case the generalized symmetry condition (6) says that the "infinitesimal deformation" $u^\alpha(x^i)+\tau G^\alpha(u^\alpha(x^i))$ is—to first order in τ—a solution of $\Xi_a=0$ whenever $u^\alpha(x^i)$ is a solution of $\Xi_a=0$.

Formally, a generalized symmetry of $\Xi_a=0$ transforms solutions into solutions: if $u^\alpha_0(x^i)$ is a solution to $\Xi_a=0$ and V is a generalized symmetry as in (7), then for any value of τ the solution $u^\alpha(x^i,\tau)$ to the Cauchy problem

$$\frac{\partial u^{\alpha}}{\partial\tau}=G^{\alpha};\qquad u^{\alpha}(x^{i},0)=u^{\alpha}_{0}(x^{i})\tag{8}$$

is also a solution to the system $\Xi_a=0$. In practice, solving (8) may be a non-trivial or impossible problem [18] but, even if this fact limits the importance of generalized symmetries for practical computations, they *are* of the greatest importance for mathematical physics, since they are an essential ingredient in Emmy Noether's theorem on the existence of conservation laws for Euler–Lagrange equations [18].

2.2 Integrability of PDEs

In modern times, research on integrability began with the discovery in 1971 (Zakharov and Faddeev, see [22] for original references) that the KdV equation (1) can be interpreted as a completely integrable hamiltonian system admitting action-angle variables, and gained impetus with the development of scattering/inverse scattering integration techniques for PDEs [1, 3, 4, 22]. It was then proven by R. Beals and D. Sattinger [5] that equations integrable via scattering/inverse scattering can be rigorously considered as infinite dimensional analogs of completely integrable hamiltonian systems. However, there are equations one would like to call integrable (e.g. Burgers' equation) that *are not* amenable to integration via scattering/inverse scattering. It is therefore quite natural to postulate a more general notion of integrability. As discussed in the Introduction, it is widely believed that the most adequate point of view is to consider generalized symmetries:

Definition 2 A partial differential equation (or a system of PDEs) is integrable (or "s-integrable") if it possesses an infinite number of nontrivial generalized symmetries.

This notion is explored for instance in [1, 6, 13–15, 17–20, 22–24, 29]. It is very remarkable that in recent years its adequacy has been corroborated a posteriori via deep classification results, see [17, 21, 27, 28].

3 On Undecidability and Incompleteness

Would it be possible to classify *all* integrable equations? (I stress once more that in this paper "integrability" means s-integrability as stated in Sect. 1 and Definition 2.) Is there an algorithm which allows one to decide whether a given equation is or is not integrable? One can prove, in fact, that it *does not* exist an algorithm able to decide whether a given partial differential equation is integrable or not, and that one cannot expect to obtain formal complete classifications.

Some notions from mathematical logic are necessary to read what follows, here only the following concepts will be (quite succinctly) recalled:

(i) A formula ϕ written in a fixed formal language *is provable from* a first order theory T (written $T \vdash \phi$), if there is a formal proof of ϕ from T.
(ii) A theory T is *complete* if and only if for every sentence ϕ either $T \vdash \phi$ or $T \vdash \neg\phi$ and, if there exists a sentence ϕ_0 such that neither $T \vdash \phi_0$ nor $T \vdash \neg\phi_0$, one says that T is *incomplete* and that ϕ_0 is *undecidable*.
(iii) The theory T is *decidable* if for any sentence ϕ one can algorithmically decide whether $T \vdash \phi$.
(iv) If $\mathcal{A}$ is a model of the fixed formal language one is using, the crucial semantic relation "$\mathcal{A}$ models or satisfies the sentence ϕ", in symbols $\mathcal{A} \models \phi$, means that indeed the sentence ϕ is true *if* it is interpreted within $\mathcal{A}$.

The reader is referred to N. da Costa and F. Doria's classic paper [7] for short explanations of these and related matters, and to the Handbook [2] for complete expositions.

3.1 The da Costa–Doria Theorem and PDEs

The main tool used in this subsection is the application of Gödel's incompleteness theorems [2, Chap. D1] to real analysis and classical mechanics developed by N. da Costa and F. Doria in [7] (see also [8, 9] and references therein). Work will be carried out within the Zermelo-Fraenkel with axiom of choice (ZFC) set theoretical framework. As it is well-known [2, Chaps. B1, B2] this framework is strong enough so that mathematical objects and relations can be represented as sets—in particular classical analysis, topology and differential geometry can be formalized within ZFC, see [9, 10] for discussions on this point—and every proof can be represented as a formal proof from the ZFC axioms.

Theorem 1

1. *One can build explicitly and algorithmically in ZFC a countable family of expressions for real-valued smooth functions* $k_m(x)$, *with* $k_m(x) \geq 0$, *such that there is no algorithm able to decide whether, for an arbitrary value of* m, *we have* $k_m(x) = 0$ *for all* $x \in \mathbf{R}$ *or not.*
2. *If* $\mathbf{M}$ *is a model of ZFC that contains the standard model of arithmetics, there exists an expression for a real smooth function* $k(x)$ *such that*

$$\mathbf{M} \models \forall x \in \mathbb{R}\, k(x) = 0$$

but

$$ZFC \nvdash \forall x \in \mathbb{R}\, k(x) = 0 \quad \text{and} \quad ZFC \nvdash \exists x \in \mathbb{R}\, k(x) \neq 0.$$

The first part of the theorem gives one undecidability; the second, incompleteness. As already explained in the Introduction, this is a deep result [7–9] whose proof is based on the classical theorem on incompleteness and undecidability of formal arithmetic by K. Gödel [2], in the undecidability of the Problem 10 of Hilbert's (there is no algorithm able to decide whether a diophantine equation is or not solvable in the integers, see [11] and [2, Chap. C.2]) and in the construction of a functor which translates diophantine equations into real functions [26].

N. da Costa and F. Doria have applied their theory to finite dimensional Hamiltonian systems [7]:

Theorem 2 *Let* (M, ω) *be a symplectic manifold of dimension* $2n$, *and let* $\{\ ,\ \}$ *be the associated Poisson bracket. Given a set of real-valued functions* $\{f_1, \dots, f_n\}$ *on* M, *there is no general algorithmic procedure that allows us to determine whether these functions are in involution* (*i.e.*, $\{f_i, f_j\} = 0$ *for all* i, j) *or not. In consequence, there is no algorithm able to check if an arbitrary finite-dimensional Hamiltonian system is integrable or not.*

In the partial differential equation case one can prove the following two results concerning symmetries:

Theorem 3 *Assume that the evolution equation* $u_t = F(x, t, u, u_x, \dots)$ *is such that* $\partial F/\partial u$ *is a smooth non-zero function of* u *and a finite number of its derivatives with respect to* x. *Then, there exists an operator* $Y = G\, \partial/\partial u$ *such that it is not possible to decide algorithmically whether* Y *is a symmetry of* $u_t = F$ *or not.*

Theorem 4 *Assume that the evolution equation* $u_t = F(x, t, u, u_x, \dots)$ *is such that* $\partial F/\partial u$ *is a smooth non-zero function of* u *and a finite number of its derivatives with respect to* x.

Then, there exists an operator $Y = G\,\partial/\partial u$ such that if **M** *is a model of ZFC that contains the standard model of arithmetics,*

$$\mathbf{M} \models Y \textit{ is a symmetry of } u_t = F$$

but

$$ZFC \nvdash Y \textit{ is a symmetry of } u_t = F$$

and

$$ZFC \nvdash Y \textit{ is not a symmetry of } u_t = F.$$

The operators Y appearing in the previous two theorems are constructed explicitly with the help of the functions $k_m(x)$ and $k(x)$ of the da Costa-Doria theorem, Theorem 1. One can strengthen Theorem 4 as follows:

Proposition 1 *Let $u_t = F(x, t, u, u_x, \dots)$ be an evolution equation and let* **M** *be as in the proposition above. There exists an expression for a vector field V on $\mathbb{R}^3$ such that*

$$\mathbf{M} \models \text{“}V \textit{ is a symmetry of } u_t = F\text{”}$$

but

$$ZFC \nvdash \text{“}V \textit{ is a symmetry of } u_t = F\text{”}$$

and

$$ZFC \nvdash \neg\text{“}V \textit{ is a symmetry of } u_t = F\text{”}.$$

Thus, symmetry theory is intrinsically incomplete and undecidable. Of course, integrability cannot fare any differently:

Theorem 5 *The theory of integrability is undecidable: there exists an evolution equation for which there is no algorithm able to assert whether it possesses a generalized symmetry or not. Thus, there is no algorithm able to assert whether it is integrable or not.*

Theorem 6 *The theory of integrability is incomplete: if* **M** *is a model of ZFC that contains the standard model of arithmetics, there exists an expression ξ in the language of ZFC such that*

$$\mathbf{M} \models \text{“The equation } u_t = \xi \text{ is integrable”}$$

but

$$ZFC \nvdash \text{“The equation } u_t = \xi \text{ is integrable”}$$

and

$$ZFC \nvdash \neg\,\text{“The equation } u_t = \xi \text{ is integrable”}.$$

The main idea behind the proofs of the last two results is to modify known integrable equations by terms depending on the functions $k_m(x)$ appearing in Theorem 1, and then to use this theorem to obtain undecidability and incompleteness.

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